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Embedding a Function into a Haar Space

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1. Introduction and Statement of Results

Let M be a set of real numbers having at least n elements and let $f(t)$ be a real valued function defined on M. Assume $n \geqslant 2$; then a sequence $S = \{x_i;\ i = 1, ..., n\}$ of elements of M is called a strong (weak) alternation of f of length n if and only if the following conditions hold.

$$x_1 < \cdots < x_n \tag{1}$$

and either $(-1)^i f(x_i)$ is positive (nonnegative) for all i, or $(-1)^i f(x_i)$ is negative (nonpositive) for all i. The same sequence S is called a strong (weak) oscillation of f of length n if and only if (1) holds and either $(-1)^i [f(x_i) - f(x_{i-1})]$ is positive (nonnegative) for $i = 2, ..., n$, or $(-1)^i [f(x_i) - f(x_{i-1})]$ is negative (nonpositive) for $i = 2, ..., n$.

Let U be an n-dimensional linear space of real valued functions defined on M and assume that M has at least $n+1$ elements. We say that U is a Haar space provided that the only element of U that has a weak alternation of length $n+1$ is the zero function. It is well known that U is a Haar space if and only if for any basis $\{f_1, ..., f_n\}$ of U, $\det[f_i(x_j);\ i, j = 1, ..., n]$ has constant (and nonzero) sign for all sets $\{x_i;\ i = 1, ..., n\}$ of points of M that satisfy (1) (cf. Zielke, [1; Lemma 3.1]). A basis of a Haar space is called a Čebyšev system.

In [2], Kurshan and Gopinath proved that if $f(t)$ is a function with a weak alternation of length n but with no weak alternation of length $n+1$, it can be embedded into an n-dimensional Haar space, i.e., that there is an n-dimensional Haar space containing f. They also raised the question of whether this result holds in the continuous case, i.e., whether if f is continuous then it can be embedded into a Haar space of continuous functions. Haverkamp and Zielke settled this question in the negative in [3] by showing that the function $g(t) = t^3[1 + (t/2) + \cos(\pi/t)]$, $t > 0$,

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$g(0)=0$, cannot be embedded into a Haar space of continuous functions on $[0, \infty)$. What makes this example even more remarkable is that $g(t)$ is continuously differentiable.

The question naturally arises as to what are the necessary and sufficient conditions for a continuous function to be embeddable into a Haar space of continuous functions. A clue can be obtained by noticing that the function $g(t)$ defined in the preceding paragraph has strong oscillations of arbitrary length. In fact, we have

THEOREM 1. *Let M be an open interval and assume that $f(t)$ is continuous in M. Then the following propositions are equivalent*:

a. *$f(t)$ can be embedded into an n-dimensional Haar space of continuous functions.*

b. *There is a strictly positive and continuous function $w(t)$ on M such that $f(t)/w(t)$ has no weak oscillation of length $n+1$ on M.*

We call $\{f_1, ..., f_n\}$ a Markov system (or a complete Čebyšev system), if and only if $\{f_1, ..., f_i\}$ is a Čebyšev system for $i=1, ..., n$. A Markov system is called normalized if and only if $f_1 \equiv 1$. (Note: Zielke ([1]) uses the terms "normed" or "1-normed.") Finally, the linear span of a (normalized) Markov system is called a (normalized) Markov space. Theorem 1 is a rather straightforward consequence of

THEOREM 2. *Let M be an interval (open, closed or semiopen, and either bounded or unbounded), and assume that $f(t)$ is continuous on M. Then the following propositions are equivalent*:

a. *$f(t)$ can be embedded into an n-dimensional normalized Markov space of continuous functions.*

b. *$f(t)$ has no weak oscillation of length $n+1$ on M.*

Remark. An inspection of the proof of Theorem 2 reveals that b is satisfied if $f(t)$ is embedded in any n-dimensional Markov space, even if not all the functions in the space are continuous.

2. PROOFS

Proof of Theorem 2. If $f(t)$ can be embedded into an n-dimensional normalized Markov space, then b follows from [1, Theorem 8.8].

Assume now that b is satisfied. It is then clear that $f(t)$ is of bounded variation in any closed subinterval of M. For any interval $[\alpha, \beta]$ contained in M, let $V(f, \alpha, \beta)$ denote the total variation of f on $[\alpha, \beta]$. Let ξ be an arbitrary but fixed point in M, and define $g(t)$ to equal $V(f, \xi, t)$ if $t \geqslant \xi$, and $-V(f, t, \xi)$ if $t<\xi$. Since $f(t)$ is continuous it is clear that also $g(t)$ is continuous. Moreover $g(t)$ must be strictly increasing, otherwise f would be constant in some interval and, since f cannot have a weak oscillation of length $n+1$, this would be a contradiction. It will thus suffice to prove the assertion for the function $v(t)=f[(g^{-1}(t)]$, whose domain is the interval $I=g(M)$ and which clearly cannot have a weak oscillation of length $n+1$.

By hypothesis, f has at most $n-2$ local extrema in (inf M, sup M), say $x_1<\cdots<x_p$. With $x_0=-\infty$ and $x_{p+1}=\infty$, let $M_i=(x_i, x_{i+1})\cap M$ for $i=0, 1, ..., p$. Wihout loss of generality let $(-1)^j f$ be strictly increasing on M_j for each j. Then v is a linear spline with knots in $g(x_1), ..., g(x_p)$ and derivative $(-1)^j$ on $g(M_j)$ for $j=0, 1, ..., p$, because for fixed k and $t, u \in M_k$ one has $v(g(u))-v(g(t))=f[g^{-1}(g(u))]-f[g^{-1}(g(t))]=f(u)-f(t)=(-1)^k\ [V(f, \xi, u)-V(f, \xi, t)]=(-1)^k\ [g(u)-g(t)]$. It is therefore easy to see that there is a polynomial $p(t)$, of degree $n-2$, such that sign $p(t)=$ sign $v'(t)$ on I, except at the points $g(x_i)$, where $v'(t)$ is undefined. Setting $w(t)=v'(t)/p(t)$ if $t\neq g(x_i)$ and $w(g(x_i))=1$, $i=1, ..., p$, we see that $v'(t)=w(t)\,p(t)$ on a set I_1 that differs from I by a finite set of points $\{x_i\}$. It is also clear that $w(t)>0$ on I_1.

Let d be an arbitrary point in I, $u_i(t)=w(t)t^i$, $y_0\equiv 1$, and for $i=1, ..., n$, $y_i(t)=\int_d^t u_{i-1}(s)\,ds$. Since $v'(t)=w(t)\,p(t)$ on I_1, $v(t)$ is clearly in the linear span of the functions y_i; thus all we need to prove is that $\{y_i;\ i=0, ..., n\}$ is a Markov system.

Note that if $\{s_i;\ i=0, ..., n\}$ is a subset of I,

$$\det[u_i(s_j);\ i, j=0, ..., k]=\left[\prod_{i=0}^{k} w(s_j)\right] V(s_0, ..., s_k),$$

where $V(s_0, ..., s_k)$ denotes the Vandermonde determinant.

Let $0\leqslant k\leqslant n$. If $\{t_i;\ i=0, ..., k\}$ is a subset of I such that $t_0<t_1<\cdots<t_k$, we have

$$\begin{aligned}
&\det[y_i(t_j);\ i, j=0, ..., k]\\
&\quad=\det[y_i(t_j)-y_i(t_{j-1});\ i, j=1, ..., k]\\
&\quad=\int_{t_0}^{t_1}\int_{t_1}^{t_2}\cdots\int_{t_{k-1}}^{t_k}\left[\prod_{i=0}^{k} w(s_i)\right] V(s_0, ..., s_k)\,ds_{k-1}\cdots ds_0,
\end{aligned}$$

whence the conclusion readily follows. Q.E.D.

Proof of Theorem 1. Assume a is satisfied and let U be an n-dimensional Haar space of continuous functions that contains f. From [4] or [5] we know hat U has a basis that is a Markov system, say $\{u_0, ..., u_{n-1}\}$.

Let $w = u_0$, $y_i = u_i/w$, and $g = f/w$. Clearly g can be embedded in the normalized Markov space spanned by the system $\{y_0, ..., y_{n-1}\}$, and b follows from Theorem 2. The converse is a trivial consequence of Theorem 2.

Q.E.D.

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